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► To cite this version:

Didier Arnal, Wissem Bakbrahem, Mohamed Selmi. QUADRATIC AND PINCZON ALGEBRAS. 2016. hal-01280499

HAL Id: hal-01280499

<https://hal.science/hal-01280499>

Preprint submitted on 1 Mar 2016

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QUADRATIC AND PINCZON ALGEBRAS

DIDIER ARNAL, WISSEM BAKBRAHEM, AND MOHAMED SELMI

ABSTRACT. Given a symmetric non degenerated bilinear form b on a vector space V , G. Pinczon and R. Ushirobira defined a bracket $\{ , \}$ on the space of multilinear skewsymmetric forms on V . With this bracket, the quadratic Lie algebra structure equation on (V, b) becomes simply $\{\Omega, \Omega\} = 0$.

We characterize similarly quadratic associative, commutative or pre-Lie structures on (V, b) by the same equation $\{\Omega, \Omega\} = 0$, but on different spaces of forms. These definitions extend to quadratic up to homotopy algebras and allows to describe the corresponding cohomologies.

1. INTRODUCTION

In [PU, DPU] (see also [D, MPU]), Georges Pinczon and Rosane Ushirobira introduced what they called a Poisson bracket on the space of skewsymmetric forms on a finite dimensional vector space V , equipped with a symmetric, non degenerated, bilinear form b . If (e_i) is a basis in V and (e'_j) the basis defined by the relations $b(e'_j, e_i) = \delta_{ij}$, the bracket is:

$$\{\alpha, \beta\} = \sum_i \iota_{e_i} \alpha \wedge \iota_{e'_i} \beta.$$

Especially, if α is a $(k+1)$ -form, and β a $(k'+1)$, then $\{\alpha, \beta\}$ is a $(k+k')$ -form. Shifting the degree on V by -1, we replace V by $V[1]$, the skew-symmetric bilinear forms on V becomes symmetric forms on $V[1]$ and the bracket $\{ , \}$ a Lie bracket.

In fact, the authors proved that a structure of quadratic Lie algebra $(V, [,], b)$ on V is completely characterized by a 3-form I , such that $\{I, I\} = 0$. The relation between the Lie bracket and I is simply $I(x, y, z) = b([x, y], z)$, and the equation $\{I, I\} = 0$ is the structure equation. A direct consequence of this construction is the existence of a cohomology on the space of forms, given by:

$$d\alpha = \{\alpha, I\}.$$

Date: March 1, 2016.

2000 Mathematics Subject Classification. 17A45, 16E40, 17BXX, 17B63.

Key words and phrases. Quadratic algebras, Cohomology, Graded Lie algebras.

This cohomology characterizes the problem of definition and deformation of quadratic Lie algebra structure on (V, b) .

In this paper, we first generalize this construction, defining the Pinczon bracket on the space $\mathcal{C}(V)$ of cyclic multilinear forms on the shifted space $V[1]$. A Pinczon bracket on $\mathcal{C}(V)$ is a Lie bracket $(\Omega, \Omega') \mapsto \{\Omega, \Omega'\}$ such that $\{\Omega, \Omega'\}$ is $(k+k')$ -linear, if Ω is $(k+1)$ -linear, and Ω' $(k'+1)$ -linear, and, for each linear form α , $\{\alpha, \ }$ is a derivation of the cyclic product.

As in [PU], there is a one-to-one correspondence between the set of Pinczon brackets on $\mathcal{C}(V)$ and the set of non degenerated, bilinear symmetric forms b on V , and the correspondence is given through the Pinczon-Ushirobira formula:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

On the other hand, it is well known that the space $\bigotimes^+ V[1]$ is a cogeбра for the comultiplication given by the deconcatenation map:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \dots \otimes x_i) \bigotimes (x_{i+1} \otimes \dots \otimes x_k).$$

Moreover, the coderivations of Δ are characterized by their Taylor series (Q_k) where $Q_k : \bigotimes^k V[1] \rightarrow V[1]$, and the bracket $[Q, Q']$ of two such coderivation is still a coderivation.

Suppose now there is a symmetric non degenerated form b on V . Denotes B the corresponding form, but on $V[1]$. There is a bijective map between $\mathcal{C}(V)$ and the space \mathcal{D}_B of B -quadratic coderivation Q , given by the formula:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

This map is an isomorphism of Lie algebra: $\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']}$.

With this construction, the notion of quadratic associative algebra, respectively quadratic associative algebra up to homotopy does coincide with the notion of Pinczon algebra structure on $\mathcal{C}(V)$. This gives also an explicit way to refine the Hochschild cohomology defined by the algebra structure.

Moreover, the subspace $\mathcal{C}_{vsp}(V)$ of cyclic, vanishing on shuffle products forms is a Lie subalgebra of $\mathcal{C}(V)$. The restriction to this subalgebra of the above construction gives us the notion of quadratic commutative algebra (up to homotopy): it is a Pinczon algebra structure on $\mathcal{C}_{vsp}(V)$. Similarly, one refines the Harrison cohomology associated to commutative algebras.

A natural quotient of $(\mathcal{C}(V), \{ , \})$ is the Lie algebra $(\mathcal{S}, \{ , \})$ of totally symmetric multilinear forms on $V[1]$. This allows us to refine the notion of quadratic Lie algebra (up to homotopy), and the corresponding Chevalley cohomology.

Considering now bi-symmetric multilinear forms on $V[1]$: *i.e.* separately symmetric in their two last variables and in all their other variables, and extending canonically the Pinczon bracket to a bracket $\{ , \}^+$ on this space of forms, we can define similarly quadratic pre-Lie algebra structures on (V, b) , and the corresponding pre-Lie cohomology.

Finally, we study the natural example of the space of $n \times n$ matrices. An unpublished preprint ([A]) contains a part of these results.

2. CYCLIC FORMS

2.1. Koszul's rule.

In this paper, V is a finite dimensional graded vector space, on a characteristic 0 field \mathbb{K} . Denote $|x|$ the degree of a vector x in V .

First recall the sign rule due to Koszul (see [Ko]). For any relation between quantities using letters representing homogeneous objects, in different orderings, it is always understood that there is an implicit $+$ sign in front of the first term. For each other term, if σ is the permutation of the letters between the first quantity and the considered term, there is the implicit sign $\varepsilon_{|letters|}(\sigma)$ (the sign of σ , taking into account only positions of the odd degree letters) in front of it.

As usual, $V[1]$ is the space V with a shifted degree. If x is homogeneous, its degree in $V[1]$ is $\deg(x) = |x| - 1$. Note simply x for $\deg(x)$ when no confusion is possible. Very generally, we use a small letter for each mapping defined on V , and capital letter for the 'corresponding' mapping, defined on $V[1]$. Let us define now these corresponding mappings.

For any k , define a ' k -cocycle' η by putting $\eta(x_1, \dots, x_k) = (-1)^{\sum_{j \leq k} (k-j)x_j}$. Then, if $\varepsilon(\sigma)$ is the sign of the permutation σ in \mathfrak{S}_k ,

$$\eta(x_{\sigma(1)}, \dots, x_{\sigma(k)})\eta(x_1, \dots, x_k) = \varepsilon(\sigma)\varepsilon_{|x|}(\sigma)\varepsilon_x(\sigma).$$

If q is a k -linear mapping from V^k into a graded vector space W , define the associated k -linear mapping from $V[1]^k$ into $W[1]$, with $\deg(Q) = |q| + k - 1$, by

$$Q(x_1, \dots, x_k) = \eta(x_1, \dots, x_k)q(x_1, \dots, x_k).$$

Preferring to keep the 0 degree for scalars, if ω is a k -linear form on V , the same formula associates to ω a form Ω , on $V[1]$, with degree $\deg(\Omega) = |\omega| + k$.

This shift of degree modifies the symmetry properties of these mappings. For instance, if q is σ -invariant, then Q is σ skew-invariant.

2.2. Pinczon bracket.

It is defined on cyclic forms.

Definition 2.1. *Let Ω be a $(k+1)$ -linear form, Ω is a cyclic form on $V[1]$, if it satisfies:*

$$\Omega(x_{k+1}, x_1, \dots, x_k) = \Omega(x_1, \dots, x_{k+1}).$$

Denote $\mathcal{C}(V)$ the space of cyclic forms on $V[1]$.

Let $\sigma \in \mathfrak{S}_k$ and Ω k -linear. Put $(\Omega^\sigma)(x_1, \dots, x_k) = \Omega(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$. Denote $Cycl$ the subgroup of \mathfrak{S}_k generated by the cycle $(1, 2, \dots, k)$. Put

$$\Omega^{Cycl} = \sum_{\tau \in Cycl} \Omega^\tau, \quad \text{and} \quad A \odot B = (A \otimes B)^{Cycl}.$$

It is easy to prove that the so defined cyclic product is commutative, but non associative.

Definition 2.2. *A Pinczon bracket $\{ , \}$ on the space $\mathcal{C}(V)$ of cyclic multilinear forms on a graded space $V[1]$ is a bilinear map such that*

1. *If \mathcal{C}^k is the space of k -linear cyclic forms, $\{\mathcal{C}^{k+1}, \mathcal{C}^{k'+1}\} \subset \mathcal{C}^{k+k'}$,*
2. *$\mathcal{C}(V)$, equipped with $\{ , \}$ is a graded Lie algebra, with center \mathcal{C}^0 ,*
3. *for any linear form α , $\{\alpha, \}$ is a derivation: if β_1, \dots, β_k are linear,*

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = \sum_j \{\alpha, \beta_j\} (\beta_1 \otimes \dots \hat{\beta}_j \dots \otimes \beta_k)^{Cycl},$$

Now,

Proposition 2.3. *1. There is a bijective map between the set \mathcal{P} of Pinczon bracket on $\mathcal{C}(V)$ and the set \mathcal{B} of degree 0, symmetric, non degenerated bilinear forms b on V ,*

2. Let b be in \mathcal{B} , and (e_i) a basis for V , if (e'_i) is the basis defined by $b(e_i, e'_j) = \delta_{ij}$, then the Pinczon bracket associated to b is:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

Proof. Let $\{ , \}$ be a Pinczon bracket on $\mathcal{C}(V)$, then $\{\mathcal{C}^0, \mathcal{C}(V)\} = 0$, and $\{\mathcal{C}^1, \mathcal{C}^1\} \subset \mathbb{K}$. The bracket defines a degree 0 bilinear, antisymmetric form B^* on $\mathcal{C}^1 = (V[1])^*$. As above this defines a symmetric bilinear form b^* on the space $V^* = (V[1])^*[1]$.

Suppose that for some α in \mathcal{C}^1 , $B^*(\alpha, \mathcal{C}^1) = \{\alpha, \mathcal{C}^1\} = 0$. Since $\{\alpha, \cdot\}$ is a derivation, $\{\alpha, \mathcal{C}^k\} = 0$, and α is a central element. Thus $\alpha = 0$, b^* is non degenerated and allows to identify V^* and V and b^* to a non degenerated bilinear symmetric form b on V .

Let B^* be a skewsymmetric bilinear form on $(V[1])^*$. For any basis (e_i) of $V[1]$ there are vectors e'_i such that:

$$B^*(\alpha, \beta) = \sum_i \iota_{e_i} \alpha \otimes \iota_{e'_i} \beta = \sum_i \alpha(e_i) \beta(e'_i) = \sum_i \iota_{e_i} \alpha \odot \iota_{e'_i} \beta.$$

Coming back to V^* this means, if all the objects are homogeneous,

$$b^*(\alpha, \beta) = \sum_i (-1)^{|\beta|} \alpha(e_i) \beta(e'_i).$$

Identify V^* to V by defining, for any γ in V^* , the vector x_γ in V such that $\alpha(x_\gamma) = b^*(\alpha, \gamma)$, for any α in V^* . Then $e'_i = x_{\epsilon_i}$, where (ϵ_i) is the dual basis of (e_i) . Therefore (e'_i) is a basis for V , and $b(e'_j, e_i) = \delta_{ij}$.

Consider now the bracket:

$$\{\Omega, \Omega'\}_P = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega',$$

where Ω is a $k+1$ -linear cyclic form and Ω' a $k'+1$ -linear one. In a following section, we shall prove this bracket defines a graded Lie algebra structure on $\mathcal{C}(V)$. It is clear that, for any α and β_j linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\}_P = \sum_j \{\alpha, \beta_j\}_P (\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_k)^{Cycl}.$$

Moreover the center of the Lie algebra $(\mathcal{C}(V), \{\cdot, \cdot\}_P)$ is \mathcal{C}^0 . In other word, $\{\cdot, \cdot\}_P$ is a Pinczon bracket.

If $k + k' \leq 0$, $\{\Omega, \Omega'\} = \{\Omega, \Omega'\}_P$. Suppose by induction this relation holds for $k + k' < N$ and consider Ω and Ω' such that $k + k' = N$. For any i ,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = -\{\Omega, \{\Omega', \epsilon_i\}_P\}_P - \{\Omega', \{\epsilon_i, \Omega\}_P\}_P = \{\epsilon_i, \{\Omega, \Omega'\}_P\}_P = \iota_{e'_i} \{\Omega, \Omega'\}_P$$

On the other hand, if $\{\Omega, \Omega'\} = \sum_\beta (\beta_1 \otimes \dots \otimes \beta_{k+k'})^{Cycl}$,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = \sum_{\beta, j} \beta_j(e'_i) (\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_{k+k'})^{Cycl} = \iota_{e'_i} \{\Omega, \Omega'\}.$$

This proves the existence and unicity, and gives the form of the Pinczon bracket associated to the symmetric, non degenerated bilinear form b on V .

□

3. CODIFFERENTIAL

3.1. General construction.

The deconcatenation Δ is a natural comultiplication in the tensor algebra $\bigotimes^+ V[1]$:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{r=1}^{k-1} (x_1 \otimes \dots \otimes x_r) \bigotimes (x_{r+1} \otimes \dots \otimes x_k).$$

The space \mathcal{D} of coderivations of Δ is a natural graded Lie algebra for the commutator. It is well known (see for instance [K, LM]) that any multilinear mapping Q can be extended in a unique way into a coderivation D_Q of Δ , and conversely any coderivation D has a unique form $D = \sum_k D_{Q_k}$. Thus the space of multilinear mappings is a graded Lie algebra for the bracket $[Q, Q'] = Q \circ Q' - Q' \circ Q$, where, if $x_{[a,b]} = x_a \otimes x_{a+1} \otimes \dots \otimes x_b$,

$$Q \circ Q'(x_{[1,k+k'-1]}) = \sum_{r=0}^{k-1} Q(x_{[1,r]}, Q'(x_{[r+1,r+k']}), x_{[r+k'+1,k+k'-1]}).$$

3.2. Relation with the Pinczon bracket.

Consider a vector space V equipped with a symmetric, non degenerated bilinear form b , with degree 0, denote B the associated form on $V[1]$. For any linear map $Q : V[1]^k \rightarrow V[1]$, define the $(k+1)$ -linear form:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}),$$

and let us say that Q is B -quadratic if Ω_Q is cyclic. Denote \mathcal{D}_B the space of B -quadratic multilinear maps.

The fundamental examples of cyclic maps are mappings associated to a Lie bracket or an associative multiplication on V . More precisely, if $(x, y) \mapsto q(x, y)$ is any internal law, with degree 0, and $Q(x, y) = (-1)^x q(x, y)$, then Ω_Q is cyclic if and only if:

$$b(q(x, y), z) = b(x, q(y, z)),$$

if and only if (V, q, b) is a quadratic algebra.

Remark that if q is a Lie bracket, then Ω_Q is symmetric. Now, if q is a commutative (and associative) product, then Ω_Q is vanishing on the image of the shuffle product on the 2 first variables:

$$sh_{(1,1)}(x_1 \otimes x_2) = (x_1 \otimes x_2) + (x_2 \otimes x_1).$$

Proposition 3.1. *The space \mathcal{D}_B of B -quadratic maps Q is a Lie subalgebra of \mathcal{D} .*

The space $\mathcal{C}(V)$ of cyclic forms, equipped with the Pinczon bracket is a graded Lie algebra, isomorphic to \mathcal{D}_B .

Proof. For any sequence $I = \{i_1, \dots, i_k\}$ of indices, denote x_I the tensor $x_{i_1} \otimes \dots \otimes x_{i_k}$.

Suppose Q is k -linear, Q' k' -linear, both B -quadratic. Thus:

$$\begin{aligned} B(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) &= -B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \\ &= B(Q(Q'(x_{[k,k+k']}), x_{[1,k-1]}), x_k). \end{aligned}$$

Therefore:

$$\begin{aligned} B(Q \circ Q'(x_{[2,k+k']}), x_1) &= \sum_{1 \leq r} B(Q(x_{[2,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k']}), x_1) \\ &= \sum_{1 \leq r < k} B(Q(x_{[1,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k'-1]}), x_{k+k'}) - B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \end{aligned}$$

Or

$$B([Q, Q'](x_{[2,k+k']}), x_1) = B([Q, Q'](x_{[1,k+k'-1]}), x_{k+k'})$$

This means $[Q, Q']$ is B -quadratic, \mathcal{D}_B is a Lie subalgebra of \mathcal{D} .

Now Ω_Q (resp. $\Omega_{Q'}$) is a $k+1$ -linear (resp. $k'+1$ -linear) cyclic form, and

$$\begin{aligned} \{\Omega_Q, \Omega_{Q'}\}(x_1, \dots, x_{k+k'}) &= \left(\sum_i \iota_{e_i} \Omega_Q \otimes \iota_{e'_i} \Omega_{Q'} \right)^{Cycl} (x_1, \dots, x_{k+k'}) \\ &= \sum_{\sigma \in Cycl} \sum_i B(Q(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}), e_i) B(Q'(x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(k+k')}), e'_i) \\ &= \sum_{\sigma \in Cycl} B(Q(x_{\sigma^{-1}([1,k])}, Q'(x_{\sigma^{-1}([k+1,k+k'])})). \end{aligned}$$

Consider a term in this sum, such that $k+k'$ belongs to $\sigma^{-1}([1,k])$. This term is:

$$B(Q(x_{[r+k'+1,k+k']}, x_{[1,r]}), Q'(x_{[r+1,r+k']})) = B(Q(x_{[1,r]}, Q'(x_{[r+1,r+k']}), x_{[r+k'+1,k+k'-1]}), x_{k+k'}).$$

The sum of all these terms is just $B((Q \circ Q')(x_{[1,k+k'-1]}), x_{k+k'})$.

Similarly, if $k+k'$ is in $\sigma^{-1}([k+1,k+k'])$, we get:

$$B(Q(x_{[r+1,r+k']}), Q'(x_{[r+k+1,k+k']}, x_{[1,r]})) = -B(Q'(x_{[1,r]}, Q(x_{[r+1,r+k]}), x_{[r+k+1,k+k'-1]}), x_{k+k'}),$$

and the corresponding sum is $-B((Q' \circ Q)(x_{[1,k+k'-1]}), x_{k+k'})$.

This proves:

$$\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']},$$

and the proposition, since $Q \mapsto \Omega_Q$ is bijective. □

Let us now study in a more detailed way different cases, when Q corresponds to an associative, or a commutative or a Lie, or a pre-Lie structure, or to an up to homotopy such structure.

4. ASSOCIATIVE PINCZON ALGEBRAS

4.1. Associative quadratic algebras.

Suppose now q is a degree 0 associative product, and b is invariant, then the associated coderivation Q of Δ , with degree 1 is the Bar resolution of the associative algebra (V, q) . The associativity of q is equivalent to the relation $[Q, Q] = 0$.

More generally, a structure of A_∞ algebra (or associative algebra up to homotopy) on the space V is a degree 1 coderivation Q of Δ on $\otimes^+ V[1]$, such that $[Q, Q] = 0$. With this last relation, the Pinczon coboundary $d_P : \Lambda \mapsto \{\Omega_Q, \Lambda\}$ is a degree 1 differential on the (graded) Lie algebra $\mathcal{C}(V)$. The corresponding cohomology is the Pinczon cohomology of cyclic forms. Write also $\Omega_{d_P Q} = d_P \Omega_Q$.

Definition 4.1. *An associative Pinczon algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 form $\Omega \in \mathcal{C}(V)$, such that $\{\Omega, \Omega\} = 0$.*

If Ω is trilinear, then an associative Pinczon algebra is simply a quadratic associative algebra (V, b, q) , where b is the symmetric non degenerated form coming from the Pinczon bracket, and q the bilinear mapping associated to Q such that $\Omega = \Omega_Q$.

Proposition 4.2. *Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be an associative Pinczon algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : \otimes^k V[1] \rightarrow V[1]$. Then Q is a structure of A_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.*

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of A_∞ algebra on V defines an unique structure of associative Pinczon algebra on V .

4.2. Bimodules and Hochschild cohomology.

Suppose (V, q) is an associative algebra and M a bi-module. Then the Hochschild cohomology with value in M is a part of the Pinczon cohomology of a natural Pinczon algebra.

Consider first the semidirect product $W = V \rtimes M$, that is the vector space $V \times M$, equipped with the associative product $q_W((x, a), (y, b)) = (q(x, y), (x \cdot b + a \cdot y))$. The dual $W^* = V^* \times M^*$ is now a (W, q_W) -bimodule, with:

$$((x, a) \cdot f)(z, c) = f((z, c)(x, a)), \quad (f \cdot (x, a))(z, c) = f((x, a)(z, c)),$$

or if $f = (g, h) \in V^* \times M^*$, $(x, a) \cdot (g, h) = (x \cdot g + a \cdot h, x \cdot h)$, $(g, h) \cdot (x, a) = (g \cdot x + h \cdot a, h \cdot x)$.

This defines a structure of algebra on the space $\tilde{V} = W \times W^*$, namely:

$$\tilde{q}((x, a, g, h), (x', a', g', h')) = (xx', xa' + ax', x \cdot g' + g \cdot x' + a \cdot h' + h \cdot a', x \cdot h' + h \cdot x'),$$

and a non degenerated symmetric bilinear form \tilde{b} ,

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

Now $(\tilde{V}, \tilde{b}, \tilde{q})$ is a quadratic associative algebra, it is the double semi-direct product of (V, q) by the bimodule M . As usual, \tilde{Q} is associated to \tilde{q} , after a shifting of degree.

Let now $c : V^k \rightarrow M$, k -linear, with degree $|c| = 2 - k$, and identify C with $\tilde{C} : \tilde{V}[1]^k \rightarrow \tilde{V}[1]$, by putting:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, g_k, h_k)) = (0, C(x_1, \dots, x_k), \sum_j C_j(x_1, \dots, h_j, \dots, x_k), 0),$$

where $C_j(x_1, \dots, h_j, \dots, x_k) = h_j(C(x_{j+1}, \dots, x_k, \cdot, x_1, \dots, x_{j-1})) \in V^*$. A direct computation shows that \tilde{C} is \tilde{B} -quadratic. A direct computation gives:

$$d_P \tilde{Q} = [\tilde{Q}, \tilde{C}] = \widetilde{[Q, C]} = \widetilde{(d_H c)[1]},$$

where d_H is the Hochschild coboundary operator on the bimodule M .

Proposition 4.3.

Let (V, q) be an associative algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear mapping c from V^k into M , with degree $2 - k$, the cyclic form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Hochschild cohomology for the (V, q) bimodule M and the Pinczon cohomology of cyclic forms $\mathcal{C}(\tilde{V})$ on \tilde{V} .

5. COMMUTATIVE PINCZON ALGEBRAS

5.1. Commutative quadratic algebras.

Consider a quadratic associative algebra (V, b, q) , but suppose now q is commutative. Consider, as above the corresponding coderivation Q . It is now anticommutative, with degree 1, and seen as a map $\underline{\otimes}^2 V[1] \rightarrow V[1]$, where $\underline{\otimes}^2 V[1]$ is the quotient of $\otimes^2 V[1]$ by the 1, 1 shuffle product $sh_{1,1}(x_1, x_2) = x_1 \otimes x_2 + x_2 \otimes x_1$.

Recall that a p, q shuffle σ is a permutation $\sigma \in \mathfrak{S}_{p+q}$ such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$. Denote $Sh(p, q)$ the set of all such shuffles. Then the p, q shuffle product on $\otimes^+ V[1]$ is

$$sh_{p,q}(x_{[1,p]}, x_{[p+1,p+q]}) = \sum_{\sigma \in Sh(p,q)} x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(p+q)}.$$

Denote $\underline{\otimes}^n(V[1])$ the quotient of $\otimes^n V[1]$ by the sum of all the image of the maps $sh_{p,n-p}$ ($0 < p < n$), and $\underline{x}_{[1,n]} = x_1 \underline{\otimes} \dots \underline{\otimes} x_n$ the class of $x_1 \otimes \dots \otimes x_n$, where the x_i

belong to $V[1]$. Finally, $\underline{\otimes}^+(V[1])$ is the sum of all $\underline{\otimes}^n(V[1])$ ($n > 0$).

On $\underline{\otimes}^+V[1]$, there is a Lie cobracket δ , defined by

$$\delta(\underline{x}_{[1,n]}) = \sum_{j=1}^{n-1} \underline{x}_{[1,j]} \otimes \underline{x}_{[j+1,n]} - \underline{x}_{[j+1,n]} \otimes \underline{x}_{[1,j]}.$$

In fact δ is well defined on the quotient and any coderivation Q of δ is characterized by its Taylor expansion $Q = \sum_k Q_k$ where each Q_k is a linear map from $\underline{\otimes}^k V[1]$ into $V[1]$ (see for instance [AAC1, BGHHW]).

Definition 5.1. *A structure of C_∞ algebra, or up to homotopy commutative algebra, on V is a degree 1 coderivation Q of δ , on $\underline{\otimes}^+V[1]$, such that $[Q, Q] = 0$.*

Associated to the notion of vanishing on shuffle products mapping Q , there is a notion of vanishing on shuffle product forms Ω .

Definition 5.2. *A $(k+1)$ -linear cyclic form Ω on the vector space $V[1]$ is vanishing on shuffle product if, for any $y, (x_1, \dots, x_k) \mapsto \Omega(x_1, \dots, x_k, y)$ is vanishing on shuffle product. Denote $\mathcal{C}_{vsp}(V)$ the space of cyclic, vanishing on shuffle product multilinear forms on $V[1]$.*

Proposition 5.3. *Suppose $\{ , \}$ is a Pinczon bracket on the space $\mathcal{C}(V)$ of cyclic multilinear forms on $V[1]$. Then $\mathcal{C}_{vsp}(V)$ is a Lie subalgebra of $(\mathcal{C}(V), \{ , \})$.*

Proof. In fact, the Pinczon bracket defines a non degenerate form b on V , thus B on $V[1]$, any form Ω can be written as $\Omega = \Omega_Q$, with

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

Therefore Ω is in $\mathcal{C}_{vsp}(V)$ if and only if Q is vanishing on shuffle products. Now, in [AAC2] it is shown that if Q, Q' are vanishing on shuffle products, then $[Q, Q']$ is also vanishing on shuffle products. This proves the proposition. \square

Definition 5.4. *A commutative Pinczon algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 form $\Omega \in \mathcal{C}_{vsp}(V)$, such that $\{\Omega, \Omega\} = 0$.*

As for associative algebra, a commutative Pinczon algebra with Ω trilinear is simply a quadratic commutative algebra (V, q, b) .

Proposition 5.5. *Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be an commutative Pinczon algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : \bigotimes^k V[1] \rightarrow V[1]$. Then Q is a structure of C_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.*

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of C_∞ algebra on V defines an unique structure of commutative Pinczon algebra on V .

5.2. (Bi)modules and Harrison cohomology.

Any module M on a commutative algebra (V, q) is a bimodule where right and left action are coinciding. Let now (V, q) be a commutative algebra, and M a (V, q) -module. Repeat the preceding construction of the quadratic associative algebra $(\tilde{V}, \tilde{b}, \tilde{q})$. Now $(\tilde{V}, \tilde{b}, \tilde{q})$ is commutative.

As above, look now for a k -linear mapping c from V^k into M , with degree $2 - k$ and vanishing on shuffle products, denote \tilde{C} the corresponding map $\tilde{V}^k \rightarrow \tilde{V}$, with degree 1, We saw that $d_P \tilde{C} = [\tilde{Q}, \tilde{C}] = [\tilde{Q}, C] = \widetilde{d_{HC}[1]}$. If we restrict this to vanishing on shuffle products map c , this is the Harrison coboundary $\widetilde{d_{Ha}c[1]}$.

Proposition 5.6. *Let (V, q) be a commutative algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear mapping c from V^k into M , with degree $2 - k$, the cyclic form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Harrison cohomology for the (V, q) bimodule M and the Pinczon cohomology of cyclic forms $\mathcal{C}(\tilde{V})$ on \tilde{V} .*

6. PINCZON LIE ALGEBRAS

6.1. Quadratic Lie algebras.

Suppose now (V, q) is a (graded) Lie algebra. Then the corresponding Bar resolution consists in replacing the space $\bigotimes^+ V[1]$ of tensors by the subspace $S^+(V[1])$ of symmetric tensors, spanned by the symmetric products $x_1 \cdot \dots \cdot x_k = \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$. Then, the natural comultiplication Δ on $S^+(V[1])$ is:

$$\Delta(x_1 \cdot \dots \cdot x_k) = \sum_{\substack{I \sqcup J = [1, k] \\ 0 < \#I < k}} x_{.I} \otimes x_{.J},$$

where, if $I = \{i_1 < \dots < i_r\}$, $x_{.I}$ means $x_{i_1} \cdot \dots \cdot x_{i_r}$.

As above, any coderivation Q of the comultiplication Δ is characterized by its Taylor coefficients $Q_k : S^k(V[1]) \rightarrow V[1]$. The bracket of two such coderivations Q ,

Q' becomes:

$$[Q, Q'](x_1 \cdot \dots \cdot x_{k+k'-1}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q(Q'(x_{\cdot J}) \cdot x_{\cdot I}) - \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#I = k}} Q'(Q(x_{\cdot I}) \cdot x_{\cdot J}),$$

Now to the Lie bracket q on V is associated a map $Q : S^2(V[1]) \rightarrow V[1]$, thus a degree 1 coderivation, still denoted Q , of Δ and the Jacobi identity for q is equivalent to the relation $[Q, Q] = 0$. The corresponding cohomology is the Chevalley cohomology.

For any k , consider $S^k(V[1])$ as a subspace of $\otimes^k V[1]$, with a projection $Sym : \otimes^k V[1] \rightarrow S^k(V[1])$:

$$Sym(x_1 \otimes \dots \otimes x_k) = x_1 \cdot \dots \cdot x_k.$$

Denote $L(\otimes^k V[1], V[1])$ the of linear maps $Q : \otimes^k V[1] \rightarrow V[1]$. By restriction to $S^k(V[1])$, Q defines a symmetric map, denote it Q^{Sym} . Then $Q^{Sym} = Q \circ Sym$, and the space $L(S^k(V[1]), V[1])$ of symmetric maps is a quotient of $L(\otimes^k V[1], V[1])$.

Similarly, the restriction of $\Omega \in \mathcal{C}^{k+1}$ is a symmetric form, denote it Ω^{Sym} : $\Omega^{Sym} = \Omega \circ Sym$. as above, the space $\mathcal{C}|_S(V)$ of the restrictions of cyclic multilinear forms to $S^+(V[1])$ is a quotient of $\mathcal{C}(V)$.

Suppose there is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, then any Ω can be written $\Omega = \Omega_Q$, with Q k -linear and B -quadratic. But, any element σ in \mathfrak{S}_{k+1} can be written in an unique way as a product $\tau \circ \rho$ where τ is in \mathfrak{S}_k , viewed as a subgroup of \mathfrak{S}_{k+1} and ρ in $Cycl$. Therefore, with our notation,

$$\Omega^{Sym} = (\Omega_Q)^{Sym} = (\Omega_{Q^{Sym}})^{Cycl} = (k+1)\Omega_{Q^{Sym}}.$$

Proposition 6.1. *The bracket defined on $L(\otimes^+ V[1], V[1])$ by the commutator of coderivations in $(\otimes^+ V[1], \Delta)$, induces a well defined bracket on $L(S^+(V[1]), V[1])$, this bracket is the above commutator of coderivations in $(S^+(V[1]), \Delta)$:*

$$[Q^{Sym}, (Q')^{Sym}] = [Q, Q']^{Sym}.$$

Any Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$ induces a well defined bracket on the quotient $\mathcal{C}|_S(V)$, this bracket denoted $\{ , \}$ is:

$$\{\Omega^{Sym}, (\Omega')^{Sym}\} = (\{\Omega, \Omega'\})^{Sym} = \frac{k+k'}{(k+1)!(k'+1)!} \sum_i \iota_{e_i}(\Omega^{Sym}) \cdot \iota_{e'_i}((\Omega')^{Sym}).$$

Proof. The first assertion is a simple computation:

$$(Q \circ Q')(x_{[1, k+k'-1]}) = \sum_{r=1}^{k-1} Q(x_{[1, r-1]} \otimes Q'(x_{[r, r+k'-1]}) \otimes x_{[r+k', k+k'-1]}).$$

Then,

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{\substack{r \\ \sigma \in \mathfrak{S}_{k+k'-1}}} Q(x_{\sigma^{-1}([1, r-1])}, Q'(x_{\sigma^{-1}([r, r+k'-1])}), x_{\sigma^{-1}([r+k', k+k'-1])})$$

Fix r and σ , put $J = \sigma^{-1}([r, r+k'-1]) = \{j_1 < \dots < j_{k'}\}$, and $j_{\tau^{-1}(t)} = \sigma^{-1}(r+t-1)$ ($1 \leq t \leq k'$). This define $\sigma \in \mathfrak{S}_{k'}$. Similarly, put $I = [1, k+k'-1] \setminus J$, $\{0\} \cup I = \{i_1 < \dots < i_k\}$ and define $\rho \in \mathfrak{S}_k$ by:

$$i_{\rho^{-1}(t)} = \sigma^{-1}(t) \quad (1 \leq t \leq r-1), \quad i_{\rho^{-1}(r)} = 0, \quad i_{\rho^{-1}(t)} = \sigma^{-1}(t-1) \quad (r+1 \leq t \leq k).$$

The correspondence $(r, \sigma) \mapsto (r, \tau, \rho)$ is one-to-one and, suming up, we get:

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q^{Sym}((Q')^{Sym}(x_{.J}) \cdot x_{.I}).$$

The commutator of coderivation of $(S^+(V[1]), \Delta)$ is the quotient bracket:

$$[Q, Q']^{Sym} = [Q^{Sym}, (Q')^{Sym}].$$

Let now $\{, \}$ be a Pinczon bracket on the space $\mathcal{C}(V)$. Thus any cyclic form Ω is written as Ω_Q . Then:

$$\{\Omega, \Omega'\}^{Sym} = \{\Omega_Q, \Omega_{Q'}\}^{Sym} = \Omega_{[Q, Q']}^{Sym} = (k+k')\Omega_{[Q, Q']^{Sym}} = (k+k')\Omega_{[Q^{Sym}, (Q')^{Sym}]}.$$

The last bracket is the commutator of coderivations in $S^+(V[1])$, thus

$$\{\Omega, \Omega'\}^{Sym}(x_{[1, k+k']}) = (k+k') \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})).$$

On the other hand,

$$\iota_{e_i} \left(\Omega_Q^{Sym} \right) (x_{[1, k]}) = (k+1) B(Q^{Sym}(x_{[1, k]}), e_i) = (k+1) (\iota_{e_i} \Omega)^{Sym} (x_{[1, k]}).$$

Therefore

$$\begin{aligned} \sum_i \iota_{e_i} \left(\Omega_Q^{Sym} \right) \cdot \iota_{e'_i} \left(\Omega_{Q'}^{Sym} \right) (x_{[1, k+k']}) &= \sum_i \left(\iota_{e_i} \left(\Omega_Q^{Sym} \right) \otimes \iota_{e'_i} \left(\Omega_{Q'}^{Sym} \right) \right)^{Sym} (x_{[1, k+k']}) \\ &= (k+1)!(k'+1)! \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})) \end{aligned}$$

This achieves the proof. \square

Explicitely, if the forms Ω and Ω' are symmetric, $\Omega^{Sym} = (k+1)!\Omega$ and the Pinczon bracket on the quotient becomes the bracket defined in [PU, DPU]:

$$\{\Omega, \Omega'\} = (k+k') \sum_i \iota_{e_i} \Omega \cdot \iota_{e'_i} \Omega'.$$

6.2. Quadratic L_∞ algebras.

As above,

Definition 6.2. *A Pinczon Lie algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 symmetric form $\Omega \in \mathcal{C}(V)^{Sym}$, such that $\{\Omega, \Omega\} = 0$.*

Since a structure of L_∞ algebra (or Lie algebra up to homotopy) on V is a degree 1 coderivation Q of $(S^+(V[1]), \Delta)$, such that the commutator $[Q, Q]$ vanishes. With the preceding computations, a Pinczon Lie algebra is in fact a quadratic L_∞ algebra.

Proposition 6.3. *Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be a Pinczon Lie algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : S^k(V[1]) \rightarrow V[1]$. Then Q is a structure of L_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.*

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of L_∞ algebra on V defines an unique structure of commutative Pinczon algebra on V .

6.3. Modules and Chevalley cohomology.

Let (V, q) be a Lie algebra, and M a (V, q) -module. To refine the corresponding Chevalley coboundary operator d_{Ch} , build, as above, the double semidirect product of V by M .

First, consider the semidirect product $W = V \rtimes M$, that is the vector space $V \oplus M$, equipped with the Lie bracket $q_W((x, a), (y, b)) = ([x, y], x \cdot b - y \cdot a)$; its dual W^* is also a (W, q_W) -module.

Then $\tilde{V} = W \times V \times W^*$ is a quadratic Lie algebra for the bracket

$$\tilde{q}((x, a, g, h), (x', a', g', h')) = ([x, x'], x \cdot a' - x' \cdot a, x \cdot g' - x' \cdot g + a \cdot h' - a' \cdot h, x \cdot h' - x' \cdot h),$$

and the non degenerated symmetric bilinear form \tilde{b}

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

A direct computation (see [BB, MR]) shows that $(\tilde{V}, \tilde{b}, \tilde{q})$ is a quadratic Lie algebra, the double semidirect product of (V, q) by its module M .

Look now for a skew-symmetric k -linear mapping c from V^k into M , with degree $|c| = 2 - k$. Associate to it the mapping:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, f_k, g_k)) = (0, C(x_1, \dots, x_k), \sum_{j=1}^k C_j(x_1, \dots, h_j, \dots, x_k), 0).$$

Clearly, \tilde{C} is totally symmetric from $\tilde{V}[1]^k$ into $\tilde{V}[1]$. More precisely $\widetilde{C^{Sym}} = \tilde{C}^{Sym}$.

Then,

$$[\tilde{Q}, \tilde{C}]^{Sym} = \widetilde{[Q, C]^{Sym}} = \widetilde{[Q, C]^{Sym}} = \widetilde{d_{Ch}c[1]}.$$

Now, if d_P is the Pinczon coboundary operator, defined by:

$$\{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\} = \Omega_{d_P \tilde{C}},$$

this can be written $d_P \tilde{C} = (2+k)\widetilde{d_{Ch}c[1]}$.

Proposition 6.4. *Let (V, q) be a Lie algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear skewsymmetric mapping c from V^k into M , with degree $2-k$, the symmetric form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Chevalley cohomology for the (V, q) module M and the Pinczon cohomology of symmetric forms $\mathcal{C}(\tilde{V})|_S$ on \tilde{V} .*

7. PINCZON PRE-LIE ALGEBRAS

7.1. Quadratic pre-Lie algebras.

A left pre-Lie algebra (V, q) is a (graded) vector space with a product q such that:

$$q(x, q(y, z)) - q(q(x, y), z) = q(y, q(x, z)) - q(q(y, x), z).$$

Then the bracket $[x, y] = q(x, y) - q(y, x)$ is a Lie bracket. Remark that any associative algebra (V, q) is a pre-Lie algebra.

A vector space M is a left (V, q) -module for the linear map $x \otimes a \mapsto x \cdot a$, if

$$q(x, y) \cdot a - x \cdot (y \cdot a) = q(y, x) \cdot a - y \cdot (x \cdot a) \quad (a \in M, x, y \in V).$$

A left module is a bi-module, if there is a linear map $a \otimes x \mapsto a * x$ such that:

$$(a * x) * y - a * q(x, y) = (x \cdot a) * y - x \cdot (a * y) \quad (a \in M, x, y \in V).$$

Then a direct computation proves

Lemma 7.1. *Let (V, q) a left pre-Lie algebra, then*

1. *The dual V^* of V is a left (V, q) -module, for: $(x \cdot f)(y) = -f(q(x, y))$.*
2. *Let M be a (V, q) bi-module then $W = V \rtimes M = (V \oplus M, q_W)$, where:*

$$q_W(x + a, y + b) = q(x, y) + x \cdot b + a * y$$

is a left pre-Lie algebra, the semi-direct product of V by M .

Chapoton and Livernet define in [CL] the notion of pre- L_∞ -algebra. If V is a graded vector space, we consider the space $S(V[1]) \otimes V([1])$, generated by the tensors

$$x_{[1,k]} \otimes y = x_1 \cdot \dots \cdot x_k \otimes y.$$

Put $P^k = S^k(V[1]) \otimes V([1])$, and $P = \sum_{k \geq 0} P^k$. On P , the coproduct Δ is defined by $\Delta(1 \otimes y) = 0$ and:

$$\Delta(x_{[1,k]} \otimes y) = \sum_{\substack{1 \leq j \leq k \\ I \sqcup J = [1,k] \setminus \{j\}}} (x_I \otimes x_j) \otimes (x_J \otimes y).$$

A linear map $Q : P^k \rightarrow P^0$ extends to a coderivation, still denoted Q by:

$$Q(x_{[1,n]} \otimes y) = \sum_{\substack{I \sqcup J = [1,n] \\ \#J=k}} x_I \otimes Q(x_J \otimes y) + \sum_{\substack{1 \leq j \leq n \\ I \sqcup J = [1,n] \setminus \{j\} \\ \#J=k}} x_I \cdot Q(x_J \otimes x_j) \otimes y.$$

The commutator of two coderivations is a coderivation, and a map $q : V \otimes V \rightarrow V$ is a left pre-Lie product if and only if the structure equation $[Q, Q] = 0$ holds for the corresponding map $Q : P^1 \rightarrow P^0$. Thus:

Definition 7.2. *A structure of pre- L_∞ algebra on V is a degree 1 coderivation Q of (P, Δ) , such that $[Q, Q] = 0$.*

Now, a quadratic pre-Lie algebra (V, b, q) is a pre-Lie algebra (V, q) equipped with a symmetric, non degenerated, degree 0, bilinear form b such that

$$b(q(x, y), z) + b(y, q(x, z)) = 0, \quad \text{or} \quad B(Q(x, y), z) = B(Q(x, z), y).$$

Example 7.3. *Let (V, q) be a left pre-Lie algebra, consider the pre-Lie algebra, semidirect product $W = V \rtimes V^*$, with*

$$q_W(x + f, y + g) = q(x, y) + x \cdot g = q(x, y) - g(q(x, \cdot)).$$

It is a quadratic pre-Lie algebra if we endow W by the canonical symmetric, non degenerated form $b(x + f, y + g) = f(y) + g(x)$.

Generalizing, let us say that a coderivation $Q = Q_0 + Q_1 + \dots$ of Δ is B -quadratic if, for any k ,

$$B(Q_k(x_{[1,k]} \otimes y_1), y_2) = B(Q_k(x_{[1,k]} \otimes y_2), y_1).$$

Then a direct computation gives:

Lemma 7.4. *Let Q and Q' be two B -quadratic coderivations of Δ . Then $[Q, Q']$ is B -quadratic.*

A structure of quadratic pre- L_∞ algebra on (V, b) is thus a B -quadratic coderivation Q of Δ such that $[Q, Q] = 0$.

7.2. Pinczon bracket for pre-Lie algebras.

To a $k + 1$ -linear coderivation Q , let us associate the form Ω_Q :

$$\Omega_Q(x_{[1,k]} \otimes y_1 \cdot y_2) = B(Q(x_{[1,k]} \otimes y_1), y_2) + B(Q(x_{[1,k]} \otimes y_2), y_1).$$

This form is separately symmetric in its k first variables, and its 2 last variables. Define thus \mathcal{P}_k the space of such forms, and $\mathcal{P}(V) = \sum_{k \geq 0} \mathcal{P}_k$. An element of $\mathcal{P}(V)$ is called a bi-symmetric form. Now it is possible to extend to $\mathcal{P}(V)$ the Pinczon bracket $\{ , \}$ associated to B . Indeed, a direct computation shows:

Lemma 7.5. *Let \mathfrak{g} be a Lie algebra and A a commutative algebra which is a right \mathfrak{g} -module such that, for any $x \in \mathfrak{g}$, $f \mapsto f \cdot x$ is a derivation of A . Then the formula:*

$$[f \otimes x, g \otimes y] = fg \otimes [x, y] + (f \cdot y)g \otimes x - f(g \cdot x) \otimes y$$

defines a Lie bracket on $A \otimes \mathfrak{g}$.

Denote $\mathcal{S}(V) = (\mathcal{C}(V))^{\text{Sym}}$ the symmetric algebra of V , it is a commutative algebra for the symmetric product \cdot . On the other hand, by construction, \mathcal{C}^2 is a Lie algebra for the Pinczon bracket $\{ , \}$, acting on \mathcal{S} through $(\Omega, \alpha) \mapsto \{\Omega, \alpha\}$. Then the properties of the Pinczon bracket assure that \mathcal{S} is a $(\mathcal{S}^2, \{ , \})$ -module and the action is a derivation of \mathcal{S} . Finally remark that $\mathcal{P}(V) = \mathcal{S}(V) \otimes \mathcal{C}^2$, therefore:

Corollary 7.6. *Let (V, b) be a vector space with a symmetric, non degenerated bilinear form, then the space $\mathcal{P}(V)$ of bi-symmetric forms on V is a Lie algebra for the Pinczon bracket:*

$$\{\Omega \otimes \alpha, \Omega' \otimes \alpha'\} = \Omega \cdot \{\alpha, \Omega'\} \otimes \alpha' + \Omega' \cdot \{\Omega, \alpha'\} \otimes \alpha + \Omega \cdot \Omega' \otimes \{\alpha, \alpha'\}.$$

This bracket is related to the commutator of B -preserving coderivations, since:

Proposition 7.7. *Suppose b is a symmetric non degenrated form on V , and Q, Q' two B -quadratic coderivations of Δ . Consider the forms $\Omega_Q, \Omega_{Q'}$ in $\mathcal{P}(V)$, then*

$$\{\Omega_Q, \Omega_{Q'}\} = 2\Omega_{[Q, Q']}.$$

Proof. Suppose $\Omega_Q = \beta \otimes \alpha \in \mathcal{P}^k$, and $\Omega_{Q'} = \beta' \otimes \alpha' \in \mathcal{P}^{k'}$, then

$$\begin{aligned} \beta \cdot \beta' \otimes \{\alpha, \alpha'\}(x_{[1, k+k']}, y_{[1, 2]}) &= \\ &= \sum_{\substack{i, I \sqcup J = [1, k+k'] \\ \#I = k}} (\Omega_Q(x_I \otimes e_i \cdot y_1) \Omega_{Q'}(x_J \otimes e'_i \cdot y_2) + \Omega_Q(x_I \otimes e_i \cdot y_2) \Omega_{Q'}(x_J \otimes e'_i \cdot y_1)) \\ &= 4 \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#I = k}} -B(Q'(x_J \otimes Q(x_I \otimes y_1)), y_2) + B(Q(x_I \otimes Q'(x_J \otimes y_1)), y_2) \end{aligned}$$

On the other hand,

$$\begin{aligned}
\beta \cdot \{\alpha, \beta'\} \otimes \alpha' (x_{[1,k+k']}, y_{[1,2]}) &= \\
&= \sum_{\substack{j, I \sqcup J = [1,k+k'] \setminus \{j\} \\ \#I=k}} \sum_i \Omega_Q(x_I \otimes e_i \cdot x_j) \Omega_{Q'}(e'_i \cdot x_J \otimes y_1 \cdot y_2) \\
&= -4 \sum_{\substack{j, I \sqcup J = [1,k+k'] \setminus \{j\} \\ \#I=k}} B(Q'(Q(x_I \otimes x_j) \cdot x_J \otimes y_1), y_2).
\end{aligned}$$

And similarly for the last term $\{\beta, \alpha'\} \cdot \beta' \otimes \alpha$ in the Pinczon bracket. Suming up, we get:

$$\begin{aligned}
\{\Omega_Q, \Omega_{Q'}\}(x_{[1,k+k']} \otimes y_{[1,2]}) &\stackrel{\deg}{=} 2B([Q, Q'](x_{[1,k+k']} \otimes y_1), y_2) + 2B([Q, Q'](x_{[1,k+k']} \otimes y_2), y_1) \\
&\stackrel{\deg}{=} 2\Omega_{[Q, Q']}(x_{[1,k+k']} \otimes y_{[1,2]}).
\end{aligned}$$

□

As for the other sort of algebras,

Definition 7.8. A Pinczon pre-Lie algebra $(\mathcal{P}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, extended to $\mathcal{P}(V)$, and a degree 3 bi-symmetric form $\Omega \in \mathcal{P}(V)$, such that $\{\Omega, \Omega\} = 0$.

Since a structure of pre- L_∞ algebra on V is a degree 1 coderivation Q of (P, Δ) , such that the commutator $[Q, Q]$ vanishes. With the preceding computations, a Pinczon pre-Lie algebra is in fact a quadratic pre- L_∞ algebra.

Proposition 7.9. Let $(\mathcal{P}(V), \{ , \}, \Omega)$ be a Pinczon pre-Lie algebra. Write $\Omega = \Omega_Q$, and $Q = \sum_k Q_k$, with $Q_k : P^k \rightarrow P^0$. Then Q is a pre- L_∞ algebra structure on V , and each Q_k is B -quadratic.

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of pre- L_∞ algebra on V defines an unique structure of Pinczon pre-Lie algebra on V .

7.3. Pinczon and pre-Lie algebra cohomologies.

Suppose (V, q) is a left pre-Lie algebra and let Q be the coderivation of Δ associated to q . Since $[Q, Q] = 0$, $d : C \mapsto [Q, C]$ is a coboundary operator.

Now, let M be a (V, q) bi-module. Let $W = V \rtimes M$ be the semi-direct product of V by M . Any map $c : \wedge^k V \otimes V \rightarrow M$ can be naturally extended to a map, still denoted c , from $\wedge^k W \otimes W$ to W . If C is the corresponding coderivation, the map dc , where the coderivation corresponding to $dc : \wedge^{k+1} W \otimes W \rightarrow W$ is $[Q, C]$, is the extension

of a map $d_{pLC} : \wedge^{k+1}V \otimes V \rightarrow M$. The operator d is the pre-Lie coboundary operator. In an unpublished work, Ridha Chatbouri computed explicitly this operator:

Proposition 7.10. *The cohomology of the pre-Lie algebra (V, q) is defined as follows. Let M be a (V, q) bi-module, the cohomology with value in M is given by the following operator: If $c : \wedge^k V \otimes V \rightarrow M$ is a $(k+1)$ -cochain, with degree $|c|$, then dc is explicitly:*

$$\begin{aligned} (-1)^{|c|} d_{pLC}(x_0 \wedge \dots \wedge x_k \otimes y) = \\ \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes x_i) * y - \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes q(x_i, y)) \\ + \sum_{i < j} (-1)^{i+j} c([x_i, x_j] \wedge x_0 \wedge \dots \hat{i} \hat{j} \dots \wedge x_k \otimes y) + \sum_{i=0}^k (-1)^i x_i \cdot c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes y). \end{aligned}$$

In [Dz], A. Dzhumaldil'daev defined a coboundary operator d for right pre-Lie algebra, which is the same as the operator computed in the preceding proposition, modulo the change of side for pre-Lie axioms. Then he used this operator to compute a corresponding homology. The proof of the proposition is the inverse of the Dzhumaldil'daev proof.

Remark 7.11. *A left (V, q) -module is nothing else than a $(V, [\ , \], q)$ -module. However the symmetry of a pre-Lie cochain differs of the symmetry of a Lie cochain. Thus the cocycles are not the same. For instance we consider $V = C_c^\infty(\mathbb{R})$, with $q(f, g) = fg'$. Choose $M = V$ and $f \cdot g = q(f, g)$. Then M is a left module. Put $c(f, g) = fg$. It is easy to verify it is a cocycle, but it is not skewsymmetric. In fact it is the coboundary of $f \mapsto b(f)$, with $(b(f))(t) = tf(t)$.*

More generally, if Q is a coderivation of Δ , which is a pre- L_∞ structure, then the operator $C \mapsto [Q, C]$ is a coboundary operator. Let us call the corresponding cohomology the (V, Q) cohomology. Then

Corollary 7.12. *Suppose (V, b, q) is a quadratic pre-Lie algebra or, more generally, $(\mathcal{P}(V), \{ \ , \ \}, \Omega)$ a Pinczon pre-Lie algebra. Then the operator $d : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ defined by $d\Omega = \{\Omega_Q, \Omega\}$ is a coboundary operator and the corresponding cohomology coincides with the (V, Q) -cohomology.*

8. A NATURAL EXAMPLE

Recall that the infinitesimal deformations of an associative (resp. Lie, resp. pre-Lie) algebra (V, q) are described by the corresponding second cohomology group

(Hochschild, pre-Lie or Chevalley) of V (see [G, NR]). Indeed, putting $q_t = q + tc$, with $t^2 = 0$, the associativity, Jacobi or pre-Lie relation are respectively equivalent to $[Q, C] = 0$. Therefore, if $t^2 = 0$, $(V + tV, q_t)$, is an associative, Lie or pre-Lie algebra if and only if c is respectively a Hochschild, a Chevalley or a pre-Lie cocycle.

Such a deformation is trivial if there is a linear map a such that $\varphi_t(x) = x + ta(x)$ satisfies $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$. With $t^2 = 0$, these conditions are equivalent to $c = d_H a$, resp. $c = d_{CH} a$, $c = d_{pL} a$. If the only infinitesimal deformations are the trivial ones, that is if the second Hochschild cohomology group $H^2((V, q))$ vanishes, we say that the corresponding structure is infinitesimally rigid.

Suppose now (V, q) is an associative algebra. Therefore it is a pre-Lie algebra for the multiplication q , and a Lie algebra for the bracket $[x, y] = q(x, y) - q(y, x)$. Some of these structures can be rigid, and other can be not rigid. Let us study here the natural associative algebra $M_n(\mathbb{K})$ of $n \times n$ matrices on a characteristic zero field \mathbb{K} . Denote $\mathfrak{gl}_n(\mathbb{K})$ the corresponding Lie algebra, and (M_n, q) the pre-Lie algebra. Remark that $\mathfrak{gl}_n(\mathbb{K})$ is a direct product: $\mathfrak{sl}_n(\mathbb{K}) \oplus \mathbb{K}id$, where $\mathfrak{sl}_n(\mathbb{K})$ is the space of traceless matrices. Put $f(x) = \frac{1}{n}\text{tr}(x)id$. Recall first some well-known results (see [CH] Chap IX.7, and use Hochschild-Serre sequence and Whitehead Lemmas [J]):

Proposition 8.1. *Let M be a $M_n(\mathbb{K})$ bi-module, then $H^k(M_n(\mathbb{K}), M) = 0$ for $k > 0$. Especially, if $M = M_n(\mathbb{K})$,*

$$H^0(M_n(\mathbb{K})) = \mathbb{K}id, \quad H^k(M_n(\mathbb{K})) = 0 \quad \text{for } k > 0.$$

Let M be a $\mathfrak{gl}_n(\mathbb{K})$ -module, and $k > 0$, then:

$$H^k(\mathfrak{gl}_n(\mathbb{K}), M) \simeq H^k(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes M^{\mathfrak{gl}_n(\mathbb{K})} \oplus H^{k-1}(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes (H^1(\mathbb{K}id, M))^{\mathfrak{gl}_n(\mathbb{K})}.$$

Especially, if $M = \mathfrak{gl}_n(\mathbb{K})$, and \tilde{f} is the class of the projection f ,

$$H^0(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}id, \quad H^1(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}\tilde{f}, \quad H^2(\mathfrak{gl}_n(\mathbb{K})) = 0.$$

Especially, $M_n(\mathbb{K})$ and $\mathfrak{gl}_n(\mathbb{K})$ are rigid. However the pre-Lie algebra V is not infinitesimally rigid:

Lemma 8.2. *The second cohomology group $H^2(M_n, q)$ of the pre-Lie algebra (M_n, q) is not vanishing.*

Proof. Following [Dz], for any matrix a , consider $c_a(x, y) = \frac{1}{n}\text{tr}(x)[y, a]$. Then:

$$d_H c_a(x, y, z) = \frac{1}{n}(\text{tr}(y)x - \text{tr}(xy)id + \text{tr}(x)y)[z, a] = (d_H f)(x, y)[z, a].$$

Then $d_{pL} c_a(x, y, z) = d_H c_a(x, y, z) - d_H c_a(y, x, z) = 0$. But remark that $Z^2(M_n(\mathbb{K})) = B^2(M_n(\mathbb{K})) = B^2(M_n, q)$. Thus, if a is not a scalar matrix, there is z such that

$[z, a] \neq 0$, and $d_H c_a(e_{12}, e_{21}, z) = -[z, a] \neq 0$, c_a is not in $B^2(M_n, q)$. \square

Let $(x, y) \mapsto q(x, y)$ any algebraic structure on a space V and $c_k : V \otimes V \rightarrow V$ ($k \geq 1$). If for any t , $q_t = q + \sum_{k>0} t^k c_k$ endows $V \otimes \mathbb{K}[[t]]$ with the same sort of structure, we say that q_t is a formal deformation of q . If there is $\varphi_t = id + \sum_{k>0} t^k b_k$, such that $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$, we say that q_t is trivial. If there is a non trivial deformation q_t , we say that (V, q) is not rigid. If moreover $c_k = 0$ for any $k > 1$, and $c_1 \notin B^2(V, q)$, we say that $q_t = q + tc_1$ is a true deformation at order 1 of (V, q) .

Using induction on k , it is easy to prove that an infinitesimally rigid structure is rigid. Of course the existence of a true deformation at order 1 implies that (V, q) is not rigid.

Corollary 8.3. *The algebras $M_n(\mathbb{K})$, $\mathfrak{gl}_n(\mathbb{K})$ are rigid. The pre-Lie algebra (M_n, q) admits true deformations at order 1.*

Proof. The first points are the previous results. Consider now the pre-Lie algebra (M_n, q) . A direct computation shows that $q_t = q + tc_a$ is a true deformation at order 1 of (V, q) (see also [Dz]). \square

Suppose now $\mathbb{K} = \mathbb{C}$. Then $x \mapsto \text{tr}(x^2)$, and $x \mapsto (\text{tr}(x))^2$ generate the space of $\mathfrak{gl}_n(\mathbb{C})$ -invariant degree 2 polynomial functions on $\mathfrak{gl}_n(\mathbb{C})$ (see [W]). Any symmetric invariant bilinear form b on $\mathfrak{gl}_n(\mathbb{C})$ can be written:

$$b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y) = \alpha \text{tr}(x(y - f(y))) + \left(\frac{1}{n}\alpha + \beta\right) \text{tr}(x)\text{tr}(y).$$

Thus, b is non degenerate, if and only if $\alpha(\alpha + n\beta) \neq 0$.

Observe now that $(x, y) \mapsto \text{tr}(xy)$ is $M_n(\mathbb{C})$ -invariant, but $(x, y) \mapsto \text{tr}(x)\text{tr}(y)$ is not $M_n(\mathbb{C})$ -invariant. Thus the non degenerated bilinear form $(x, y) \mapsto \text{tr}(xy)$ generates the cone of $M_n(\mathbb{C})$ -invariant symmetric bilinear forms.

Similarly, if b is (M_n, q) -invariant, it is $\mathfrak{gl}_n(\mathbb{C})$ -invariant, thus $b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y)$, and the invariance relation reads:

$$\alpha \text{tr}(xyz + yxz) + \beta(\text{tr}(xy)\text{tr}(z) + \text{tr}(y)\text{tr}(xz)) = 0.$$

Choosing for instance $x = y = z = e_{11}$, we get $2(\alpha + \beta) = 0$, then $x = y = z = id$ gives $2n(\alpha + n\beta) = 0$, the unique (M_n, q) -invariant symmetric bilinear form is identically zero. Summarizing,

Lemma 8.4. 1. *The associative algebra $M_n(\mathbb{C})$ is a quadratic algebra for the 1-dimensional cone of symmetric bilinear forms $\alpha \text{tr}(xy)$,*

2. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is a quadratic Lie algebra for the 2-dimensional cone of symmetric bilinear forms $\alpha \operatorname{tr}(xy) + \beta \operatorname{tr}(x)\operatorname{tr}(y)$ ($\alpha(\alpha + n\beta) \neq 0$),
3. The pre-Lie algebra (M_n, q) is not a quadratic pre-Lie algebra.

Recall that in Example 7.3, we saw that the space $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ is a quadratic pre-Lie algebra for the product and the bilinear form:

$$q_W(x_1 + x_2, y_1 + y_2) = x_1 y_1 - y_2 x_1, \quad b(x_1 + x_2, y_1 + y_2) = \operatorname{tr}(x_2 y_1 + x_1 y_2).$$

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